Discrete Applied Mathematics 2 (1980) 155–158

NOTE

THE NUMBER OF ADDITIONAL VARIABLES REQUIRED FOR THE INTEGER PROGRAMMING FORMULATION

Toshihide IBARAKI

Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, Kyoto, Japan

Received 21 February 1979
Revised 16 November 1979

It is known that a minimization problem having a finite feasible region with k elements can be formulated as an integer programming problem by introducing at most $\lceil \log_2 k \rceil$ additional integer variables. In this note, we show that this bound is best possible in the sense that some minimization problem actually requires $\lceil \log_2 k \rceil$ additional variables.

1. Introduction

The formulation of a minimization problem as an integer programming (abbreviated to IP) problem has been discussed in various papers such as Meyer [3], Ibaraki [1] and Jeroslow [2]. A minimization problem:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in S, \tag{1}$$

where $f: S \to \mathbf{R}$ and $S \subset \mathbf{R}^n$ ($\mathbf{R}$ denotes the set of real numbers) is said to have an IP formulation if there exist positive integers m and p, a vector $b \in \mathbf{R}^p$ and a $(m+n+1) \times p$ matrix A such that

$$\begin{aligned} &x \in S \text{ and } z = f(x) \text{ hold if and only if there exist } x \in \mathbf{R}^n \\ &\text{and } z \in \mathbf{R} \text{ satisfying } (x, z, y)A \leq b \text{ for some } y \in \mathbf{Z}^m, \end{aligned} \tag{2}$$

where $\mathbf{Z}$ denotes the set of integers.

Integer variables $y_1, y_2, \ldots, y_m$ are *additional variables* introduced for the IP formulation. The above definition is a slightly modified version of [1], in that [1] uses the all-integer formulation (i.e., $S \subset \mathbf{Z}^n$, $f(x) \in \mathbf{Z}$ and $x \in \mathbf{Z}^n$ are assumed in (1) and (2)) whereas the present one is the mixed-integer formulation. A formulation which is more general than the above mixed-integer formulation is adopted in [3].

It can be easily shown by extending the proof in [1] (corresponding to the slight difference in the definition) that if S is a finite set with k elements, then a minimization problem (1) can be formulated as an IP problem with at most $m=\lceil\log_2 k\rceil$ additional variables ($\lceil X\rceil$ denotes the smallest integer not smaller than X). Reference [1] also conjectures that the bound is best possible in the sense that some minimization problems actually require at least $\lceil\log_2 k\rceil$ additional variables.

In this note, we prove this conjecture affirmatively.

2. Proof of the conjecture

It will be shown through several lemmas that the following problem requires at least $\lceil\log_2 k\rceil$ additional variables.

$$\begin{aligned}&S=\{(x_1,x_2,\ldots,x_n)\mid x_j\in\{0,1\},\quad j=1,2,\ldots,n\}\quad(\subset\mathbf{R}^n)\\&f(x)=0\quad\text{for } x\in S.\end{aligned}\tag{3}$$

Note that S is a finite set with $k=2^n$ elements and $\lceil\log_2 k\rceil=n$. All the elements in S are extreme points of Conv S, where Conv X denotes the convex hull of set X.

Assume that an IP formulation of this problem with the minimum number of additional variables is obtained. Define for $x\in\mathbf{R}^n$

$$\psi(x)=\{y\in\mathbf{Z}^m\mid(\exists z\in\mathbf{R})((x,z,y)A\leqslant b)\}.\tag{4}$$

Condition (2) asserts that $\psi(x)\neq\emptyset$ if and only if $x\in S$. For problem (3) we can say more.

Lemma 1. *$\psi(x^a)\cap\psi(x^b)=\phi$ holds for $x^a\neq x^b\in S$.*

Proof. If $y'\in\psi(x^a)\cap\psi(x^b)$, then (x^a,z^a,y') and (x^b,z^b,y') are feasible solutions of $(x,z,y)A\leqslant b$, where $z^i=f(x^i)$. Then the convex sum

$$\lambda(x^a,z^a,y')+(1-\lambda)(x^b,z^b,y')$$

is also a feasible solution for any $0\leqslant\lambda\leqslant1$. This implies

$$\lambda x^a+(1-\lambda)x^b\in S$$

by definition (2), but it is a contradiction to the definition (3) of S, in which each $x\in S$ is an extreme point of Conv S.

Lemma 2. *Let $Y=\bigcup_{x^i\in S}\psi(x^i)$. ($Y$ has at least $k=2^n$ points by Lemma 1.) Let* Ext X *denote the set of extreme points of a convex set X. Then*

(i) Conv $Y\cap Z^m=Y$ (i.e., Conv Y *does not contain any integer point other than those in* Y).

(ii) *If* $y^0 \in \psi(x^a)$ *satisfies* $y^0 \notin \operatorname{Ext} \operatorname{Conv} Y$, *i.e.*,

$$y^0 = \sum_{y \in \operatorname{Ext}\operatorname{Conv} Y, y \neq y^0} \lambda_y y, \tag{5}$$

where $0 \leq \lambda_y \leq 1$ *and* $\sum \lambda_y = 1$, *then* $\lambda_y = 0$ *holds unless* $y \in \psi(x^a)$.

(iii) $\operatorname{Ext} \operatorname{Conv} Y \cap \psi(x^i) \neq \emptyset$ *for any* $x^i \in S$.

Proof. (i) Consider an integer point $y^0 \in \operatorname{Conv} Y \cap \mathbf{Z}^m$ such that

$$y^0 = \sum_{y \in Y} \lambda_y y,$$

where $0 \leq \lambda_y \leq 1$ and $\sum \lambda_y = 1$. Then $(\sum_{i=1}^k \lambda_i x^i, \sum_{i=1}^k \lambda_i z^i, y^0)$ is obviously a feasible solution of $(x, z, y)A \leq b$, where $z^i = f(x^i)$ and $\lambda_i = \sum_{y \in \psi(x^i)} \lambda_y$ for $x^i \in S$. Recall here that, if λ_i were not zero or one, $\sum_{i=1}^k \lambda_i x^i \notin S$ follows from definition (3) of S. If λ_1 are zero or one (i.e., exactly one λ_i is one and others are zero), however,

$$y^0 = \sum_{y \in \psi(x^a)} \lambda_y y \tag{6}$$

holds for some $x^a \in S$. This implies $y^0 \in \psi(x^a)$, i.e., $y^0 \in Y$, as easily shown.

(ii) By the same reasoning as in (i) (replacing Y by $\operatorname{Ext}\operatorname{Conv} Y$), we can conclude from $y^0 \in \psi(x^a)$ that $\lambda_a = 1$ and $\lambda_i = 0$ for all $i \neq a$. Therefore $\lambda_y = 0$ holds in (5) unless $y \in \psi(x^a)$.

(iii) $\psi(x^i) \neq \emptyset$, property (ii) and Lemma 1 immediately imply (iii).

The following lemma is cited in [4] with a credit to J.W.S. Cassels. The proof is rather simple and included here for completeness.

Lemma 3. *Let* $\bar{Y} \subset \mathbf{R}^m$ *be a closed convex set with* $\operatorname{Ext} \bar{Y}\ (=\{y^1, y^2, \ldots, y^k\}) \subset \mathbf{Z}^m$ *which is finite. If* $\bar{Y} \cap \mathbf{Z}^m = \operatorname{Ext} \bar{Y}$, *then* $k \leq 2^m$ *must hold.*

Proof. If $k > 2^m$, some $i_1 \neq i_2$ satisfy $y^{i_1} \equiv y^{i_2} \pmod 2$ since each component of y is either even or odd. Then $y' = \frac{1}{2}(y^{i_1} + y^{i_2})$ is an integer point not in $\operatorname{Ext} \bar{Y}$. This is a contradiction.

As a consequence of the above lemmas, we have the next theorem.

Theorem 1. *A minimization problem* (3) *requires at least* $n = \lceil \log_2 k \rceil$ *additional variables when it is formulated as an IP problem in the sense of* (2).

Proof. Consider an IP formulation (2) with the minimum number of additional variables and let Y^* denote the projection of the feasible region of $(x, z, y)A \leq b$ on the y-sapce, i.e.,

$$Y^* = \{y \in \mathbf{R}^m \mid (\exists x \in \mathbf{R}^n, z \in \mathbf{R})((x, z, y)A \leq b\} \quad (\subset \mathbf{R}^m).$$

By condition (2), $Y^* \cap \mathbf{Z}^m = Y$ holds, where Y is defined in Lemma 2. Now, for each $x^i \in S$, delete from (Ext Conv $Y) \cap \psi(x^i)$ $(\subset Y)$ all the points except one, and denote the set Y after this modification by Y'. Y' satisfies

$$|\text{Ext Conv } Y' \cap \psi(x^i)| = 1 \tag{7}$$

for all $x^i \in S$, where $|\cdot|$ denotes the cardinality of the set therein. This is possible by property (iii) of Lemma 2. Then we have

$$\text{Conv } Y' \cap \mathbf{Z}^m = \text{Ext Conv } Y', \tag{8}$$

as shown below. First Conv $Y' \cap \mathbf{Z}^m \supset$ Ext Conv Y' is obvious. To prove the converse, consider $y^0 \in$ Conv $Y' \cap \mathbf{Z}^m$. Then $y^0 \in \psi(x^a)$ holds for some $x^a \in S$ by $Y' \subset Y$ and property (i) of Lemma 2, and hence

$$y^0 = \sum_{y \in (\text{Ext Conv } Y') \cap \psi(x^a)} \lambda_y y,$$

where $0 \leq \lambda_y \leq 1$ and $\sum \lambda_y = 1$, by Ext Conv $Y' \subset$ Ext Conv Y and property (ii) of Lemma 2. However, this and (7) imply $y^0 \in$ Ext Conv Y', proving

$$\text{Conv } Y' \cap \mathbf{Z}^m \subset \text{Ext Conv } Y'.$$

Now let $\bar{Y} =$ Conv Y' in Lemma 3. Since $Y^* \supset$ Conv Y' is obvious by definition and Conv Y' has $k = 2^n$ extreme points by (7), we can conclude that $2^m \geq k$, i.e., $m \geq \lceil \log_2 k \rceil = n$ by Lemma 3.

Remark. It is not known whether problem (3) requires at least $n = \lceil \log_2 k \rceil$ additional variables if the all-integer definition [1] is used. In the case of Meyer's mixed-integer definition [3], however, the above proof can be directly extended to show that at least $\lceil \log_2 k \rceil$ additional integer variables are necessary.

Acknowledgments

The author wishes to thank Professors H. Mine and T. Hasegawa of Kyoto University for their support. He is also indebted to Professor R.G. Jeroslow of Georgia Institute of Technology for having informed him of reference [4].

References

[1] T. Ibaraki, Integer programming formulation of combinatorial optimization problems, Discrete Math. 16 (1976) 39–52.

[2] R.G. Jeroslow, Representations of unbounded optimizations as integer programs, Management Science Research Report No. 389, Carnegie-Mellon University, April 1976.

[3] R.R. Meyer, Integer and mixed-integer programming models: General properties, J. Optimization Theory Appl. 16 (1975) 191–206.

[4] H.E. Scarf, An observation on the structure of production sets with indivisibilities, Proc. Nat. Acad. Sci. U.S.A. 74 (1977) 3637–3641.